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Surface tension in Potts models and percolation

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Abstract. The Kasteleyn-Fortuin relation is used to give a simple proof for the s-state Potts model of the relation $\beta\sigma\xi = 1$ between the correlation length in a particular direction on a planar lattice and the surface tension of an interface with corresponding orientation on the dual, thus generalising a well known result for the Ising model. The relation is also used to give an interpretation of the surface tension in the percolation problem in general dimensions.

1. Introduction

The representation of the Potts model partition function due to Kasteleyn and Fortuin (1969) and Fortuin and Kasteleyn (1972) has been used to extend the model to general values of s, the number of states in the original model. Of particular importance has been the correspondence between the limit $s \to 1$ and the bond percolation problem (see, e.g., Lubensky 1979). The Kasteleyn-Fortuin representation has also been used, in conjunction with simple results from graph theory, to derive duality relations connecting the partition function at high temperatures on a planar lattice with the low-temperature partition function on the dual lattice (Wu 1978, 1982). In this paper, it is generalised to treat Potts models on lattices with fixed state boundary conditions, in terms of which the surface tension can be defined (see Rottman and Wortis (1984) for a general discussion). On taking the limit $s \to 1$ in § 3, a physical interpretation of surface tension in the bond percolation problem is obtained and used to establish the result stated in the abstract for $s = 1$. The general result is proved in § 4 and the paper is concluded with a discussion in § 5. The next section comprises definitions and general results to be used in subsequent sections.

2. General theory

We shall work throughout with the following Potts Hamiltonian and partition function:

$$-\beta\mathcal{H} = Ks \sum_{(ij)} (\delta_{\sigma_i\sigma_j} - 1) + Hs \sum_i (\delta_{\sigma_i 1} - 1) \tag{1}$$

$$Z = s^{-N} \sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H}). \tag{2}$$

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The state variables σ_i in (1) take the values 1, 2, ..., s and the first summation on the RHS denotes a sum over nearest-neighbour pairs of lattice sites. The coupling K is positive and the quantity $\delta_{\sigma\sigma'}$ is unity if $\sigma = \sigma'$, zero otherwise. We discuss the applicability of the results of this paper to more general Hamiltonians than (1) in § 5.

When the number of lattice sites N is infinite, the model defined by (1) and (2) undergoes a phase transition at a finite coupling K_c, in zero field H, for dimensionality $d > 1$. The transition may be characterised by the order parameter $s\langle\delta_{\sigma_i 1}\rangle - 1$ where $\langle\ldots\rangle$ denotes the usual thermal average. Explicitly

$$\lim_{H\to 0^+} s\langle\delta_{\sigma_i 1}\rangle - 1 = 0 \qquad K < K_c \tag{3a}$$

$$\lim_{H\to 0^+} s\langle\delta_{\sigma_i 1}\rangle - 1 > 0 \qquad K > K_c. \tag{3b}$$

Substituting (1) into (2), and rearranging, we may write the Potts partition function in the form

$$Z = s^{-N} \sum_{\{\sigma_i\}} \left(\prod_{(ij)} (1 - p + p\delta_{\sigma_i\sigma_j}) \right) \exp\left(Hs \sum_i (\delta_{\sigma_i 1} - 1) \right) \tag{4}$$

where

$$p = 1 - e^{-sK}.$$

Next we expand out the factor $\Pi_{(ij)}(1 - p + p\delta_{\sigma_i\sigma_j})$ in (4) and associate each term in the expansion with a graph G on the lattice. Each graph is a configuration of occupied and unoccupied bonds such that if $\delta_{\sigma_i\sigma_j}$ is present in the corresponding term the bond between i and j is occupied and is otherwise unoccupied. Weight factors p and $1 - p$ are associated with occupied and unoccupied bonds respectively.

We define $\mathcal{N}(G, n_b, n_s) \equiv$ number of n_b (bond), n_s (site) clusters of occupied bonds per site of the lattice in G (isolated sites count as 0-bond, 1-site clusters). When the sum over $\{\sigma_i\}$ is performed in the Potts partition function, each cluster will yield a non-vanishing contribution only if all the sites within it carry the same spin label. Bearing this in mind we have

$$Z = s^{-N} \sum_G p^{\mathcal{N}_B(G)} (1-p)^{N_B - \mathcal{N}_B(G)} \prod_{n_s, n_b} [1 + (s-1)\exp(-sHn_s)]^{N\mathcal{N}(G, n_b, n_s)}. \tag{5}$$

Here

$$\mathcal{N}_B(G) = \sum_{n_s, n_b} n_b N\mathcal{N}(G, n_b, n_s) = \text{total number of occupied bonds in } G$$

where N_B is the total number of nearest-neighbour bonds in the lattice.

The result (5) is the Fortuin and Kasteleyn (1972) representation of the Potts partition function. The simple derivation given above follows that of Lubensky (1979). Setting $H = 0$ in (5), we obtain the result

$$Z = s^{-N} \langle s^{\mathcal{N}_c(G)} \rangle_G \tag{6}$$

where

$$\langle\ldots\rangle_G = \sum_G p^{\mathcal{N}_B(G)} (1-p)^{N_B - \mathcal{N}_B(G)}$$

$$\mathcal{N}_c(G) = \sum_{n_s, n_b} N\mathcal{N}(G, n_b, n_s) = \text{total number of clusters in } G.$$

Turning now to the order parameter and correlation function, we define the quantities

$$\mathcal{N}_c(G, a) \equiv \text{number of clusters in } G \text{ containing site } a.$$

$$\mathcal{N}_c(G, a, b) \equiv \text{number of clusters in } G \text{ containing sites } a \text{ and } b.$$

Then

$$\langle \delta_{\sigma_a 1} \rangle = s^{-N} Z^{-1} \sum_{\{\sigma_i\}} \delta_{\sigma_a 1} \prod_{(ij)} (1 - p + p\delta_{\sigma_i \sigma_j})$$

$$= \langle s^{\mathcal{N}_c(G) - \mathcal{N}_c(G,a)} \rangle_G / \langle s^{\mathcal{N}_c(G)} \rangle_G. \tag{7}$$

Clearly

$$\mathcal{N}_c(G, a) = 1$$

so that

$$\langle \delta_{\sigma_a 1} \rangle = 1/s. \tag{8}$$

We remark that (8) is consistent with (3) for $K < K_c$ only. This is because, when $K > K_c$, contributions to $\langle \delta_{\sigma_a 1} \rangle$ of graphs containing infinite clusters are non-zero. The presence of an infinitesimal positive field in (5) forces these clusters to be in state 1. Thus for an infinite system in the presence of such a field, (6) holds provided that $\mathcal{N}_c(G)$ refers to finite clusters only. Similarly (7) will hold provided that $\mathcal{N}_c(G, a)$ refers to finite clusters only and therefore this quantity can be 1 or 0. Thus the general result for an infinite system in an infinitesimal positive field is

$$\langle \delta_{\sigma_a 1} \rangle \geqslant 1/s.$$

Using (7) and assuming (3), it is easily deduced that the relative contribution to $\langle \delta_{\sigma_a 1} \rangle$ and hence to Z from graphs containing infinite clusters which account for a finite fraction of the sites is zero for $K < K_c$ and a finite quantity for $K > K_c$.

We shall restrict ourselves to $K < K_c$ and $H = 0$, and consider the correlation function

$$C_{ab} = \langle (s\delta_{\sigma_a 1} - 1)(s\delta_{\sigma_b 1} - 1) \rangle / (s - 1). \tag{9}$$

Applying the relation

$$\sum_{\alpha} \delta_{\sigma_b \alpha} = 1$$

to the RHS of (9) and using (8), we obtain

$$C_{ab} = 1 - s^2(s-1)^{-1} \sum_{\alpha \neq 1} \langle \delta_{\sigma_a 1} \delta_{\sigma_b \alpha} \rangle = 1 - s^2 \langle \delta_{\sigma_a 1} \delta_{\sigma_b 2} \rangle. \tag{10}$$

Now

$$\langle \delta_{\sigma_a 1} \delta_{\sigma_b 2} \rangle = s^{-N} Z^{-1} \sum_{\{\sigma_i\}} \delta_{\sigma_a 1} \delta_{\sigma_b 2} \prod_{(ij)} (1 - p + p\delta_{\sigma_i \sigma_j}). \tag{11}$$

Because the Potts states at sites a and b are constrained to be different in the numerator of the RHS of (11), terms in the corresponding graphical expansion for which a and b are members of the same cluster contribute zero on performing the sum over states. Hence

$$\langle \delta_{\sigma_a 1} \delta_{\sigma_b 2} \rangle = s^{-N} Z^{-1} \langle \delta_{\mathcal{N}_c(G,a,b),0} s^{\mathcal{N}_c(G) - \mathcal{N}_c(G,a) - \mathcal{N}_c(G,b)} \rangle_G$$

$$= s^{-(N+2)} Z^{-1} \langle \delta_{\mathcal{N}_c(G,a,b),0} s^{\mathcal{N}_c(G)} \rangle_G. \tag{12}$$

Substituting (12) into (10), we may write

$$C_{ab} = s^{-N} Z^{-1} \langle (1 - \delta_{\mathcal{N}_c(G,a,b),0}) s^{\mathcal{N}_c(G)} \rangle_G = \langle s^{\mathcal{N}_c(\tilde{G})} \rangle_{\tilde{G}} / \langle s^{\mathcal{N}_c(G)} \rangle_G \tag{13}$$

where the graphs $\tilde{G}$ are those for which a and b are connected by a cluster of occupied bonds. Taking the limit $s \to 1$ and interpreting p as a bond occupation probability, we deduce that

$\lim_{s\to 1} C_{ab}$ = probability that sites a and b are connected in the bond percolation problem

$= C_{ab}(p)$, the correlation function for bond percolation.

Restricting ourselves to two dimensions for the reminder of this section, we consider a convex region of an infinite lattice which is divided into two parts by a line of length L lattice spacings. We vary L by varying the lattice spacing, keeping the orientation of the lattice fixed with respect to the figure, so that the angle ϕ between the line and some lattice axis remains fixed. For each L, the sites a and b are chosen to be the lattice sites inside the region, nearest the two points of intersection of the line with the boundary of the region. This procedure is shown in figure 1 for a triangular lattice. We define the correlation length ξ in the direction ϕ as

$$\xi^{-1}(\phi, s) \equiv \lim_{L\to\infty} [-L^{-1} \ln(C_{ab})]. \tag{14}$$

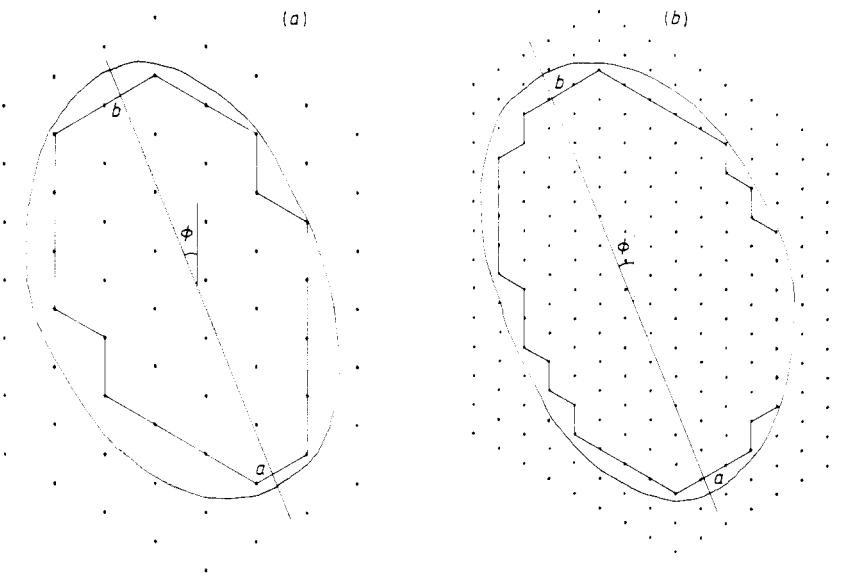

Figure 1. Illustrating the procedure, described in the text, for defining the correlation length $\xi(\phi, s)$.

Equation (14) is just the usual definition of the correlation length as the exponential decay length of the correlation of distant states. We also define a correlation function C'_{ab} on the finite portion of the lattice enclosed within the region (shown shaded in figure 1). C'_{ab} is given by

$$C'_{ab} = \langle s^{\mathcal{N}_c(\tilde{G})} \rangle_{\tilde{G}} / \langle s^{\mathcal{N}_c(G)} \rangle_G$$

where $\tilde{G}$, G are now graphs on the finite lattice (cf 13),

$$C'_{ab}(p) = \lim_{s\to 1} C'_{ab}$$

is the bond percolation correlation function between the sites a and b on the finite lattice. We assume that

$$\lim_{L\to\infty}[-L^{-1}\ln(C'_{ab})]=\xi^{-1}(\phi, s). \tag{15}$$

Equation (15) is the strongest assumption we shall make in deriving the duality relations which are the main object of this paper. Physically, the result is obvious. When L is large C_{ab} is very small. This is because the contribution from graphs containing a cluster large enough to connect a and b is a very small fraction of the total contribution to Z from all graphs. The bulk of this very small contribution will come from graphs where the cluster connecting a and b is not much bigger than it has to be—in other words, is very long and thin. The graphs G on the infinite lattice can be divided into sets which yield the same graph on the finite lattice. For graphs $\tilde{G}$, the typically long thin cluster which connects a and b on the infinite lattice may wander outside the region and therefore possibly be broken on the finite lattice. Thus not all $\tilde{G}$ on the infinite lattice yield $\tilde{G}$ on the finite lattice and C'_{ab} is strictly less than C_{ab}. Intuitively we expect that, as $L\to\infty$, C'_{ab} will tend to a finite fraction of C_{ab}, or, at worst, some inverse power of L times C_{ab}. In either case, the assumption (15) will be valid.

3. Surface tension and bond percolation

Consider the Potts model defined on a d-dimensional infinite lattice and consider a convex region of the lattice. Let this region be divided into two parts by a $(d-1)$-dimensional plane where the 'area' of intersection with the region is A in units of (lattice spacing)$^{(d-1)}$ and whose normal is in direction $\hat{n}$ with respect to the lattice axes. We vary A by changing the lattice spacing, while keeping $\hat{n}$ fixed. The situation is just the d-dimensional analogue of figure 1. For a given A we consider the finite lattice of complete lattice cells, together with their vertices and bonds, contained in the region. The boundary spins can be divided into two sets C_1 and C_2, according to which side of the dividing plane they are on. We define

$$Z^{\mu\nu}=\sum_{\{\sigma_i\}}\left(\prod_{\substack{i\in C_1\\ j\in C_2}}\delta_{\sigma_i\mu}\delta_{\sigma_j\nu}\right)\exp(-\beta\mathscr{H}). \tag{16}$$

Thus $Z^{\mu\nu}$ is the finite lattice partition function with the boundary spins C_1 and C_2 fixed in the states μ and ν, respectively. We use the following standard definition for the surface tension σ of an interface normal to $\hat{n}$:

$$\sigma(\hat{n}, s)=\lim_{A\to\infty}A^{-1}[-\beta^{-1}\ln(Z^{12})+\beta^{-1}\ln(Z^{11})]. \tag{17}$$

As usual, we assume that the limit on the RHS of (17) is independent of the shape of the bounding region. Let us define the following quantities

$\mathcal{N}_c(G, C_k)$ = number of clusters in G containing at least one site of the set C_k, where $k=1, 2$.

$\mathcal{N}_c(G, C_1, C_2)$ = number of clusters in G containing at least one site from each of the sets C_1 and C_2.

Then, clearly,

$$Z^{12} = s^{-N}\langle \delta_{\mathcal{N}_c(G,C_1,C_2),0} s^{\mathcal{N}_c(G)-\mathcal{N}_c(G,C_1)-\mathcal{N}_c(G,C_2)}\rangle_G \tag{18}$$

$$Z^{11} = s^{-N}\langle s^{\mathcal{N}_c(G)-\mathcal{N}_c(G,C_1)-\mathcal{N}_c(G,C_2)+\mathcal{N}_c(G,C_1,C_2)}\rangle_G. \tag{19}$$

Substituting (18) and (19) into (16) and taking the limits $s \to 1$, we obtain the result

$$\sigma(\hat{n}, 1) = \lim_{A\to\infty} [-A^{-1}\beta^{-1} \ln\{Q(A, p)\}] \tag{20}$$

where

$$Q(A, p) = \langle \delta_{\mathcal{N}_c(G,C_1,C_2),0}\rangle_G. \tag{21}$$

In the language of the bond percolation problem, the quantity on the RHS of (21) is the probability that there is no cluster of occupied bonds connecting the set of boundary spins C_1 to the set of boundary spins C_2. Let us now specialise to two dimensions and consider an infinite planar lattice and its dual. We consider a convex region, divided into two parts by a line of length L, and the finite portion of $\mathcal{L}$ of the original lattice contained within it as we did in the introduction. $\mathcal{L}$ will contain sites a and b selected by the procedure described in the introduction. We now consider a finite portion $\mathcal{L}^*$ of the dual lattice which consists of all bonds which cross bonds of $\mathcal{L}$ and the vertices attached to these bonds. The $\mathcal{L}^*$ corresponding to the $\mathcal{L}$ of figure 1(a) is shown in figure 2. We note that $\mathcal{L}^*$ is *not* the dual of $\mathcal{L}$. The boundary spins are divided into two sets C_1 and C_2 by the continuation of the line. Denoting the Potts model partition function on $\mathcal{L}^*$ with coupling K^* by $Z_*(K^*)$, the surface tension of an interface at angle ϕ for a Potts model with coupling K^* on a lattice dual to the original infinite planar lattice is given by

$$\sigma^*(\phi, s) = \lim_{L\to\infty} L^{-1}[-(\beta^*)^{-1} \ln Z_*^{12}(K^*) + (\beta^*)^{-1} \ln Z_*^{11}(K^*)]. \tag{22}$$

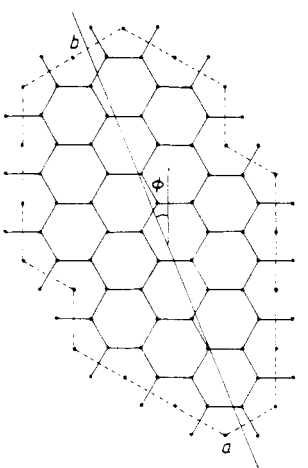

Figure 2. The lattice $\mathcal{L}^*$ corresponding to the lattice $\mathcal{L}$ of figure 1(a) (shown dotted).

With each graph G on $\mathcal{L}$ may be associated a unique graph G^* on $\mathcal{L}^*$ such that each occupied bond of G crosses an unoccupied bond of G^* and vice versa. It is easy to see that G^* for which C_1 and C_2 are connected correspond to G for which a and b

are not connected and vice versa. Then

$$\delta_{\mathcal{N}_c(G^*,C_1,C_2),0} = 1 - \delta_{\mathcal{N}_c(G,a,b),0}.$$

Further, since

$$\mathcal{N}_B(G^*) = N_B - \mathcal{N}_B(G)$$

we deduce that

$$\langle \delta_{\mathcal{N}_c(G^*,C_1C_2),0} \rangle_{G^*} = \langle 1 - \delta_{\mathcal{N}_c(G,a,b),0} \rangle_G \tag{23}$$

provided that we make the identification

$$p^* = 1 - p. \tag{24}$$

The quantity on the RHS of (23) is the bond percolation correlation function $C'_{ab}(p)$. Using (20), (21) and assuming (15), we obtain the result that

$$\beta^* \sigma^*(\phi, 1) \xi(\phi, 1) = 1. \tag{25}$$

Equation (25), in conjuction with (24), is a special case of a corresponding result for general s that we shall prove in the next section.

4. Proof of the duality relation for general s

The method of attack we shall use to generalise (24) and (25) will be to establish a relation between the partition function on $\mathcal{L}^*$, in terms of which the surface tension is defined, and that on $\mathcal{L}^{(1)}$, the lattice dual to $\mathcal{L}$, so that standard duality arguments can be used. $\mathcal{L}^{(1)}$ is obtained from $\mathcal{L}^*$ by coalescing the boundary sites C_1 and C_2 into a single site C, usually called the exterior site. We denote the partition function, with coupling K^*, defined on $\mathcal{L}^{(1)}$ by $Z_{(1)}(K^*)$. Similarly, it is conveneint to define a lattice $\mathcal{L}^{(2)}$ which is derived from $\mathcal{L}^*$ by coalescing C_1 into a single site C_1 and C_2 into a single site C_2. If C_1 is fixed in state μ and C_2 is fixed in state ν we denote the partition function on $\mathcal{L}^{(2)}$ by $Z^{\mu\nu}_{(2)}(K^*)$. There is a one-to-one correspondence between the bonds of any pair of the three lattices $\mathcal{L}^*$, $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ and hence G^* describes a graph on any of the lattices. For any such graph, the number of internal clusters, i.e. clusters containing no sites C_1, C_2 or C, is independent of the lattice. Denoting this quantity by $\mathcal{N}_I(G^*)$, we have

$$\mathcal{N}_I(G) = \mathcal{N}_c(G) - \mathcal{N}_c(G, C_1) - \mathcal{N}_c(G, C_2) + \mathcal{N}_c(G, C_1, C_2). \tag{26}$$

The terms on the RHS of (26) are lattice-dependent. (For $\mathcal{L}^{(1)}$ we understand $C_1 = C_2 = C$ on the RHS of (26).)

Let N^*, $N^{(1)}$ and $N^{(2)}$ denote the number of sites of $\mathcal{L}^*$, $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$, respectively. Then

$$N^{(1)} = N^{(2)} - 1 = N^* - n_c + 1 \tag{27}$$

where

$$n_c = \text{number of boundary sites of } \mathcal{L}^*.$$

Applying the general result (19) to the lattice $\mathcal{L}^*$ we have

$$Z^{11}_*(K^*) = s^{-N} \langle s^{\mathcal{N}_I(G^*)} \rangle_{G^*}.$$

Using (27), (10), and the fact that, on lattice $\mathscr{L}^{(1)}$,

$$\mathcal{N}_{\mathrm{I}}(G^*) = \mathcal{N}_{\mathrm{c}}(G^*) - 1 \tag{28}$$

we obtain the result that

$$Z^{11}_*(K^*) = s^{n_{\mathrm{c}}} Z_{(1)}(K^*). \tag{29}$$

We note that (29) could have easily been established for $s = 2, 3, 4, \ldots$, by starting from the original model (1) and (2). The proof given, starting from (6) and (27), holds for general s. Furthermore, on noting that $\mathcal{N}_{\mathrm{c}}(G^*, C_1, C_2)$ is zero on $\mathscr{L}^{(2)}$ if and only if it is zero on $\mathscr{L}^*$, we use the result (18) to deduce that

$$Z^{12}_*(K^*) = s^{-(n_{\mathrm{c}}-2)} Z^{12}_{(2)}(K^*). \tag{30}$$

We can write

$$Z^{12}_{(2)}(K^*) = s^{-N^{(2)}} \langle s^{\mathcal{N}_{\mathrm{I}}(\tilde{G}^*)} \rangle_{\tilde{G}^*} \tag{31}$$

where the $\tilde{G}^*$ are those graphs on $\mathscr{L}^{(2)}$ for which there is no cluster of occupied bonds, connecting the sites C_1 and C_2. The lattice $\mathscr{L}^{(1)}$, dual to $\mathscr{L}$, must be constructed so that the exterior site lies in the region exterior to $\mathscr{L}$. We then see that, on $\mathscr{L}^{(1)}$, $\tilde{G}^*$ are graphs such that the exterior site contains no circuit which crosses the line ab. Considering the RHS of (31) on $\mathscr{L}^{(1)}$ and using (28), we have

$$Z^{12}_{(2)}(K^*) = s^{-(N^{(1)}+2)} \langle s^{\mathcal{N}_{\mathrm{c}}(\tilde{G}^*)} \rangle_{\tilde{G}^*}. \tag{32}$$

Substituting (29), (30) and (32) into (22), and using (6) we obtain the following expression for the surface tension:

$$\sigma^*(\phi, s) = \lim_{L\to\infty} [-L^{-1}(\beta^*)^{-1} \ln\{\langle s^{\mathcal{N}_{\mathrm{c}}(\tilde{G}^*)} \rangle_{\tilde{G}^*} / \langle s^{\mathcal{N}_{\mathrm{c}}(G^*)} \rangle_{G^*}\}]. \tag{33}$$

The expression on the RHS of (33) involves graphs and cluster numbers on $\mathscr{L}^{(1)}$ and hence we can apply standard duality arguments. The basic result is (see, e.g., Wu 1982)

$$(p^*)^{\mathcal{N}_{\mathrm{B}}(G^*)} (1-p^*)^{N_{\mathrm{B}} - \mathcal{N}_{\mathrm{B}}(G^*)} s^{\mathcal{N}_{\mathrm{c}}(G^*)} = A[p^{\mathcal{N}_{\mathrm{B}}(G)} (1-p)^{N_{\mathrm{B}} - \mathcal{N}_{\mathrm{B}}(G)} s^{\mathcal{N}_{\mathrm{c}}(G)}]. \tag{34}$$

The relation (53) holds for any graph G^* on $\mathscr{L}^{(1)}$ and its dual G on $\mathscr{L}$, where A is a constant which depends on p and $\mathscr{L}$, but not on G, provided that the following holds:

$$p^*/(1-p^*) = s(1-p)/p. \tag{35}$$

Equation (35) is usually written in the form

$$(\mathrm{e}^{sK} - 1)(\mathrm{e}^{sK^*} - 1) = s. \tag{36}$$

Applying (34) to the RHS of (33) we have

$$\langle s^{\mathcal{N}_{\mathrm{c}}(\tilde{G}^*)} \rangle_{\tilde{G}^*} / \langle s^{\mathcal{N}_{\mathrm{c}}(G^*)} \rangle_{G^*} = \langle s^{\mathcal{N}_{\mathrm{c}}(\tilde{G})} \rangle_{\tilde{G}} / \langle s^{\mathcal{N}_{\mathrm{c}}(G)} \rangle_{G}. \tag{37}$$

It is easily seen that any graph $\tilde{G}$ dual to $\tilde{G}^*$ is a graph in which a and b are connected by a cluster of occupied bonds and vice versa. Thus we identify the RHS of (37) as the correlation function C'_{ab}. Substituting (37) into (33) with this identification and assuming (15), we deduce that

$$\beta^* \sigma^*(\phi, s) \xi(\phi, s) = 1 \qquad K < K_{\mathrm{c}}. \tag{38}$$

Equation (38), in conjunction with (36), is the main result of this paper and generalises the well known result for $s = 2$ (Watson 1968).

5. Discussion

Firstly, we shall make a few remarks about the result (38), which, in spite of the simplicity of its derivation, does not seem to have appeared previously in the literature. Earlier work on surface tension in general spin systems, in which duality arguments were used, has been carried out by mathematicians concerned with using it to establish a rigorous criterion for the existence of a phase transition in such systems (Fontaine and Gruber 1979). The surface tension of an interface parallel to a lattice axis has been calculated for the three-state Potts model on a square lattice by a modified solid-on-solid approach (Selke and Pesch 1982). The Potts surface tension also makes an appearance in droplet theories on the Potts model (Lubensky and McKane 1981, Schmittmann 1982). In particular, if $\psi(V)\,\mathrm{d}V$ is the fraction of the system volume occupied by droplets (with non-overlapping boundaries) of volume $V \to V + \mathrm{d}V$ the following result holds:

$$\lim_{V\to\infty} \ln \psi(V) = -\oint_{S_{\min}} \mathrm{d}S\, \sigma^*(\hat{n}, s) \qquad K^* > K_c^* \tag{39}$$

where $\hat{n}$ is the normal to the surface element $\mathrm{d}S$ and $S_{\min}$ is the closed surface which minimises the RHS of (39) subject to the constraint that it bounds a region of fixed volume V and describes the equilibrium droplet shape. For the special case of the square lattice Ising model an exact expression for this shape has been obtained using (38) in conjunction with the exactly known $\xi(\phi)$ (Zia and Avron 1982). No corresponding exact results are available for other values of s or for other lattices. Let us specialise to a general planar lattice and $s < 4$, so that the system undergoes a continuous phase transition. At criticality σ^* becomes isotropic and the equilibrium droplet shape tends to a circle. Two-scale factor universality (Ferer and Wortis 1972) and duality arguments may be applied to show that, at criticality

$$\xi/\xi^* = \kappa_0^-/\kappa_0^+. \tag{40}$$

The quantity on the RHS of (40) is the universal amplitude ratio of subcritical to supercritical inverse correlation length amplitudes. Using (38) and (40), (39) becomes at criticality,

$$\lim_{V\to\infty} \ln \psi(V) = -2\pi^{1/2}(\kappa_0^+/\kappa_0^-)\, V^{1/2}/\xi^*.$$

Bruce (1984) has presented a simple argument that

$$\lim_{K^*\to(K_c^*)^-} \beta^*\sigma^*\xi^* = \tfrac{1}{2} \qquad d = 2,\ s < 4. \tag{41}$$

Taken in conjunction with (40) and (38), (41) implies the result

$$\kappa_0^-/\kappa_0^+ = 2.$$

Let us now comment on the physical interpretation of the surface tension in the bond percolation problem furnished by equations (20) and (21). In particular we shall briefly discuss the consequences of adopting (20) and (21) as the *definition* of the surface tension in the bond percolation problem, so that it can be considered independent of any Potts model construction. The properties that $\sigma(\hat{n}, 1)$ is non-negative and is zero in the disordered phase easily follow from this definition, the first of these properties simply being a consequence of the fact that $Q(A, p)$, being a probability, is less than

or equal to unity, while the second follows from the expectation that, in the disordered phase, where the system contains only finite clusters, $Q(A, p)$ will tend to a constant as $A \to \infty$. Furthermore, for $p > p_c$, the result (39) for $s = 1$ can be derived starting from (20) and (21), leading to the physical interpretation of $\psi(V)\,\mathrm{d}V$, for $s = 1$ and large V, as the fraction of the system volume occupied by regions, with non-overlapping boundaries, of volume $V \to V + \mathrm{d}V$ such that no site in the interior of any region is connected to any site outside the region. As $V \to \infty$ the fraction of such regions containing a cluster of n_s sites, and vice versa, such that

$$\lim_{V\to\infty} n_s / V = \rho P(p) \tag{42}$$

tends to unity. Here ρ is the number of lattice sites per unit volume and $P(p)$ is the percolation probability. This intuitive observation allows us to deduce the result that

$$\lim_{V\to\infty} \ln C(n_s)/\ln \psi(V) = 1 \tag{43}$$

where n_s and V are related by (42). $C(n_s)$ is the number of clusters per site of the lattice containing n_s sites. Using (43) and (42) in (39) with $s = 1$, we can derive an expression for the large n_s behaviour of $\ln C(n_s)$. The result is the same as that obtained by using the relation between $C(n_s)$ and the formal one-state limit of the s-state Potts model free energy in a field, and considering droplet solutions of the latter (Harris and Lubensky unpublished, Lubensky and McKane 1981). The fact that we can derive this result starting from (20) and (21) and using simple arguments entirely in the language of the bond percolation problem suggests that this approach may also be useful in obtaining a physical picture of other features of the Potts model droplet theory applied to percolation. Before moving on to discuss generalisations of (38), we remark that a surface tension has been defined for the site percolation problem by exploiting the analogy between cluster perimeters and surface energy in the Ising model (Franke 1982a, b).

It would be interesting to investigate the connections between a surface tension defined in this way and using the present boundary condition approach, bearing in mind the correspondence between site percolation and the $s \to 1$ limit of a Potts model with multisite interactions (Giri *et al* 1977).

Lastly, we consider generalisations of (38). We need not restrict ourselves to nearest-neighbour interactions in (1) provided that bonds between sites where interactions are present form a planar graph. Furthermore, the result generalises to bond-dependent interactions $K_{ij} > 0$, since the argument of § 4 is the same up to equation (33) while (34) and (35) are replaced by the corresponding 'local' duality relations (for a discussion see Wu 1978).

In conclusion, we have given a simple derivation, under rather weak assumptions, of a duality relation involving the Potts model surface tension on a planar lattice and the correlation function on the dual. We have also provided a consistent physical interpretation of surface tension in the bond percolation problem.

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